Worksheet 17 Solutions - MA114 Summer 18

1) a) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^3+1}$. Apply LCT with $\sum_{n=1}^{\infty} \frac{1}{n}$.

$$c = \lim_{n\to\infty} \frac{\frac{n^2-1}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3-n}{n^3+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{1+\frac{1}{n^3}} = 1.$$

Since $0<c=1<\infty$, the LCT tell's us both series converge or diverge together. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty}\frac{n^2-1}{n^3+1}$.

b) $\sum_{k=1}^{\infty} \frac{k\ln k}{(k+1)^3}$. Use the fact that $\ln(k)<\sqrt{k}$. ~~for large k~~

Then $0<\frac{k\ln k}{(k+1)^3} < \frac{k\sqrt{k}}{(k+1)^3} < \frac{k\sqrt{k}}{k^3} = \frac{1}{k^{3/2}}$. Since $\sum_{k=1}^{\infty}\frac{1}{k^{3/2}}$ is a convergent p-series, the series converges by the Comparison Test.

c) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$. $\ln(n)>1$ for $n\geq 3$, so $\frac{\ln(n)}{n} > \frac{1}{n}$ for $n\geq 3$. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ by the Comparison Test.

d) $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$. Use the ratio test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\frac{n!}{3^n n^2}\right| = \lim_{n\to\infty} 3\cdot\frac{1}{n+1}\cdot\frac{(n+1)^2}{n^2} = \lim_{n\to\infty}\frac{3(n+1)}{n^2} = \lim_{n\to\infty}\frac{3\left(\frac{1}{n}+\frac{1}{n^2}\right)}{1} = 0$$

Since $L=0<1$, the series converges (absolutely) by the Ratio Test.

e) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot(2n-1)}{2\cdot 5\cdot 8\cdot\ldots\cdot(3n-1)}$. Use the ratio test.

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)\cdot(2n+1)}{2\cdot5\cdot8\cdot\ldots\cdot(3n-1)\cdot(3n+2)}\cdot\frac{2\cdot5\cdot8\cdot\ldots\cdot(3n-1)}{1\cdot3\cdot5\cdot\ldots\cdot(2n-1)}\right| = \lim_{n\to\infty}\frac{2n+1}{3n+2} \overset{L'H}{=} \lim_{n\to\infty}\frac{2}{3} = \frac{2}{3}$$

Since $L=\frac{2}{3}<1$, the series converges by the ratio test.

f) $\sum_{n=1}^{\infty}\left(\frac{1}{n^3}+\frac{1}{3^n}\right)$. This series is the sum of a convergent p-series $(p=3)$ and a convergent geometric series $(r=\frac{1}{3})$, so it converges.

g) $\sum_{k=1}^{\infty} k^5 \sqrt[3]{k^{-21}} = \sum_{k=1}^{\infty} k^5 (k^{-21})^{\frac{1}{3}} = \sum_{k=1}^{\infty} k^5 k^{-7} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series ($p=2>1$).

h) $\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{\sqrt{n}}$. Use AST, since the series is alternating. $b_n = \frac{\ln(n)}{\sqrt{n}}$

1) Is $b_n \to 0$? $\lim_{n\to\infty} \frac{\ln(n)}{\sqrt{n}} \overset{L'H}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = \lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0.$

2) Is $b_{n+1} < b_n$ for all n?

$f(x) = \frac{\ln(x)}{\sqrt{x}}$. Then $f'(x) = \frac{\frac{1}{x}\sqrt{x} - \ln(x)\frac{1}{2\sqrt{x}}}{x} = \frac{1}{x\sqrt{x}} - \frac{\ln(x)}{2x\sqrt{x}} = \frac{2-\ln(x)}{2x\sqrt{x}} < 0$ if $\ln(x) > 2$

i.e. if $x > e^2 \approx 7.39$. So $b_{n+1} < b_n$ for all $n \geq 8$.

Therefore by the AST the series converges.

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2) a) The integral test cannot be applied to this series because $|\cos(x)| + \frac{1}{x}$ is not a decreasing function.

b) $\sum_{n=1}^{\infty} r^n = \sum_{n=0}^{\infty} r^{n+1} = \sum_{n=0}^{\infty} r \cdot r^n$ converges to $\frac{r}{1-r}$.

c) False. For example, $\sum_{n=1}^{\infty} \frac{1}{n}$.

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3) a) $\lim_{n\to\infty} \frac{(-1)^n}{2\sqrt{n}} = 0$ by the Squeeze Theorem: $-\frac{1}{2\sqrt{n}} \leq \frac{(-1)^n}{2\sqrt{n}} \leq \frac{1}{2\sqrt{n}}$ and $\lim_{n\to\infty} -\frac{1}{2\sqrt{n}} = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0$.

b) $\lim_{n\to\infty} \frac{4^n}{1+9^n} = \lim_{n\to\infty} \frac{\left(\frac{4}{9}\right)^n}{\left(\frac{1}{9}\right)^n + 1} = \frac{0}{0+1} = 0.$

c) $\lim_{n\to\infty} \cos\left(\frac{2n+1}{3n^2}\pi\right) = \cos\left(\lim_{n\to\infty} \frac{2n+1}{3n^2}\pi\right) = \cos(0) = 1$

d) $\lim_{n\to\infty} \ln(n+1) - \ln(en+2) = \lim_{n\to\infty} \ln\left(\frac{n+1}{en+2}\right) = \ln\left(\lim_{n\to\infty} \frac{n+1}{en+2}\right) = \ln\left(\frac{1}{e}\right) = -1.$

4) a) $\sum_{n=0}^{\infty} \frac{x^n}{n^4+2}$. $\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)^4+2} \cdot \frac{n^4+2}{x^n} \right| = \lim_{n\to\infty} \frac{n^4+2}{(n+1)^4+2} \cdot |x| = \lim_{n\to\infty} \frac{1+\frac{2}{n^4}}{(1+\frac{1}{n})^4 + \frac{2}{n^4}} |x| = \frac{1+0}{(1+0)^4+0}|x| = |x| < 1.$

So $\boxed{R=1}$. I.O.C.: $|x|<1 \iff -1<x<1$, So test $x=-1$, $x=1$.

$x=1$: $\sum_{n=0}^{\infty} \frac{1}{n^4+2}$ converges by comparison to $\sum_{n=0}^{\infty} \frac{1}{n^4}$. $x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^4+2}$ converges by AST.

So I.O.C. is $\boxed{[-1,1]}$.

b) $\sum_{n=\phi}^{\infty} \frac{(x-1)^{3n+2}}{\ln(n)}$. $L=\lim_{n\to\infty}\left|\frac{(x-1)^{3n+5}}{\ln(n+1)}\cdot\frac{\ln(n)}{(x-1)^{3n+2}}\right| = \lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}|x-1|^3 \overset{LH}{=} \lim_{n\to\infty}\frac{n+1}{n}|x-1|^3 = |x-1|^3 < 1.$

So $|x-1|<1$, hence $\boxed{R=1}$. If $|x-1|<1$, $-1<x-1<1$, so $0<x<2$.

Test $x=0$, $x=2$:

$x=0$: $\sum_{n=1}^{\infty}\frac{(-1)^{3n+2}}{\ln(n)} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\ln(n)}$ converges by AST.

$x=2$: $\sum_{n=1}^{\infty}\frac{1^{3n+2}}{\ln(n)} = \sum_{n=1}^{\infty}\frac{1}{\ln(n)}$ diverges by comparison to $\sum_{n=1}^{\infty}\frac{1}{n}$ since $\ln(n)<n$, so $\frac{1}{n}<\frac{1}{\ln(n)}$.

So I.O.C. is $[0,2)$.

c) $\sum_{n=0}^{\infty}\frac{(-5)^n}{n!}(x+10)^n$

$L=\lim_{n\to\infty}\left|\frac{(-5)^{n+1}(x+10)^{n+1}}{(n+1)!}\cdot\frac{n!}{(-5)^n(x+10)^n}\right| = \lim_{n\to\infty}\left|\frac{-5(x+10)}{n+1}\right| = 0<1$ for all x.

So $R=\infty$ and I.O.C. is $(-\infty,\infty)$.

6) $f_{avg} = \frac{1}{2-1}\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx$

$u=\frac{\pi}{x}$ $du = -\frac{\pi}{x^2}dx$ $x=1: u=\pi$; $x=2: u=\pi/2$

$f_{avg} = \int_{\pi}^{\pi/2}\frac{\sin(u)}{-\pi}\,du$

$= \frac{1}{\pi}\int_{\pi/2}^{\pi}\sin(u)\,du$

$= -\frac{1}{\pi}\cos(u)\Big|_{\pi/2}^{\pi}$

$= -\frac{1}{\pi}(\cos(\pi)-\cos(\pi/2))$

$= \boxed{\frac{1}{\pi}}$

5) Know that $\sum_{n=0}^{\infty}x^n = \frac{1}{1-x}$ on $(-1,1)$.

- $\frac{x^3}{5+x^2} = x^3\cdot\frac{1}{5+x^2} = x^3\cdot\frac{1}{5}\cdot\frac{1}{1+\frac{x^2}{5}} = \frac{x^3}{5}\cdot\frac{1}{1-\left(-\frac{x^2}{5}\right)} = \frac{x^3}{5}\sum_{n=0}^{\infty}\left(-\frac{x^2}{5}\right)^n$ if $\left|-\frac{x^2}{5}\right|<1$

$= \frac{x^3}{5}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{5^n}$ if $|x^2|<5$

$= \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{5^{n+1}} \text{ if } |x|<\sqrt{5}}$

- $\arctan(x) = \int\frac{1}{1+x^2}dx = \int\frac{1}{1-(-x^2)}dx = \int\sum_{n=0}^{\infty}(-x^2)^n\,dx = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \text{ if } |x^2|<1\text{, i.e. if } |x|<1}$

- $\frac{2}{(1-x)^2} = 2\cdot\frac{1}{(1-x)^2} = 2\cdot\frac{d}{dx}\left(\frac{1}{1-x}\right) = 2\cdot\frac{d}{dx}\left(\sum_{n=0}^{\infty}x^n\right)$ if $|x|<1$

$= 2\sum_{n=0}^{\infty}nx^{n-1}$

$= \boxed{\sum_{n=0}^{\infty}2(n+1)x^n \text{ if } |x|<1}$